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A Note on Functions with Interlacing Roots

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INTRODUCTION

In the investigation of optimal Lagrange interpolation [3–5] and in other treatments of optimal interpolation [1–3], two results on the linear independence of sets of polynomials with interlacing roots have been used. It is the purpose of this note to record generalizations of these results with new proofs. Though these two results were stated for sets of polynomials, they would apply in a much wider context. Proposition 1 below corresponds to Lemma 8 of [4] and [5], while Proposition 2 below improves upon Lemma 9 of [5].

NOTATION

We assume $T_1, ..., T_n$ are points (real numbers) with

$$T_1 < T_2 < \cdots < T_n,$$

and that $q_1, ..., q_n$ are functions which lie in the span of an $(n-1)$-dimensional Tchebycheff system.

We assume further that $q_1, ..., q_n$ are such that, for $i \in \{1, ..., n\}$ and for $j \in \{1, ..., n-1\}$, q_i has exactly one root in each of the subintervals (T_j, T_{j+1}) of $[T_1, T_n]$, except that q_i has no root in (T_i, T_{i+1}) for $i \in \{1, ..., n-1\}$, nor in (T_{i-1}, T_i) for $i \in \{2, ..., n\}$. We further assume that $q_i(T_j) \neq 0$ for $i, j \in \{1, ..., n\}$.

RESULTS

PROPOSITION 1. *For $p \in \{1, ..., n\}$, any set $\{q_1, ..., q_n\} - \{q_p\}$ of functions described above is linearly independent.*

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Proposition 2. *For $k, p \in \{1,..., n\}$, $k \neq p$, no non-trivial linear combination of the functions $\{q_1,..., q_n\} - \{q_k, q_p\}$ may have roots in the same subintervals as the roots of q_k or q_p.*

Proof of Proposition 1. Let a linear combination

$$\sum_{j=1}^{n} a_j q_j$$

be given which is identically zero, and in which, for some specified p,

$$a_p = 0.$$

It suffices to show that all of the coefficients are zero. No generality is lost by assuming that $p \neq 1$, since, in that case, the indices and order relations in the proposition may be reversed and read from right to left. Moreover, no generality is lost by assuming that $a_1 \geqslant 0$, and that

$$q_j(T_1) > 0 \qquad \text{for} \quad j \in \{1,..., n\}. \tag{1}$$

The sign changes of $q_1,..., q_n$ assumed in the proposition, then, imply that

$$\operatorname{sgn} q_j(T_i) = (-1)^{i+1} \qquad \text{for} \quad i, j \in \{2,..., n\}, i \neq j, \tag{2}$$

$$\operatorname{sgn} q_i(T_i) = (-1)^{i} \qquad \text{for} \quad i \in \{2,..., n\} \tag{3}$$

$$\operatorname{sgn} q_1(T_i) = (-1)^{i} \qquad \text{for} \quad i \in \{2,..., n\}. \tag{4}$$

Statements (2), (3), and (4) together imply (5).

$$q_j(T_i)\, q_1(T_i) < 0 \qquad \text{for} \quad i, j \in \{2,..., n\}, j \neq i. \tag{5}$$

Let

$$\mathscr{S} = \{j\colon j \neq 1 \text{ and } a_j \geqslant 0\}, \text{ and } \mathscr{R} = \{1,..., n\} - \mathscr{S}.$$

Then $\mathscr{S}$ is not empty because $p \in \mathscr{S}$, and $\mathscr{R}$ is not empty, since $1 \in \mathscr{R}$. We also define

$$S = \sum_{j \in \mathscr{S}} a_j q_j \qquad \text{and} \qquad R = \sum_{j \in \mathscr{R}} a_j q_j.$$

Clearly, $S + R = 0$, $S(T_1) \geqslant 0$, and

$$R(T_1) \leqslant 0. \tag{6}$$

If $i \in \mathscr{S}$, then by (5), $q_j(T_i)\, q_1(T_i) < 0$ for all $j \in \mathscr{R}$, $j \neq 1$, whence, since $a_1 \geqslant 0$,

$$a_j q_j(T_i)\, q_1(T_i) \geqslant 0 \qquad \text{for all } j \in \mathscr{R}.$$

Thus,

$$R(T_i)\, q_1(T_i) = \sum_{j \in \mathscr{R}} a_j q_j(T_i)\, q_1(T_i) \geqslant 0, \qquad \text{for} \quad i \in \mathscr{S}. \tag{7}$$

If $i \in \mathscr{R}$, $i \neq 1$, then by (5) again, $q_j(T_i)\, q_1(T_i) < 0$, for $j \in \mathscr{S}$, whence

$$a_j q_j(T_i)\, q_1(T_i) \leqslant 0 \qquad \text{for} \quad j \in \mathscr{S}.$$

Thus,

$$R(T_i)\, q_1(T_i) = -S(T_i)\, q_1(T_i) = -\sum_{j \in \mathscr{S}} a_j q_j(T_1)\, q_1(T_i) \geqslant 0. \tag{8}$$

By (4), (6), (7), and (8), therefore,

$$(-1)^i\, R(T_i) \geqslant 0 \qquad \text{for} \quad i \in \{1,..., n\}$$

whence, $R = 0$ and $S = 0$. But, if $S = 0$, then $a_j = 0$ for all $j \in \mathscr{S}$, for otherwise $S(T_1) > 0$. Hence, $a_j \leqslant 0$ for all $j \in \{2,..., n\}$.

Now, by (5), as above, $q_j(T_p)\, q_1(T_p) < 0$ for $j \neq 1, p$. Thus, since $a_p = 0$ and $a_1 \geqslant 0$,

$$a_j q_j(T_p)\, q_1(T_p) \geqslant 0 \qquad \text{for all } j \in \{1,..., n\}$$

and

$$0 = \sum_{j=1}^{n} a_j q_j(T_p) = \sum_{j=1}^{n} a_j q_j(T_p)\, q_1(T_p),$$

and the expression on the right is a sum of non-negative terms. It follows that $a_j = 0$ for all $j \in \{1,..., n\}$.

Proof of Proposition 2. This follows as a corollary of Proposition 1. Let

$$q_k' = \sum_{\substack{j=1 \\ j \neq p}}^{n} a_j q_j$$

be a function with roots in the same subintervals as those of q_k. Then

$$\{q_1,..., q_{k-1}, q_k', q_{k+1},..., q_n\}$$

is a set of functions obeying the hypotheses of Proposition 1. By Proposition 1, therefore, the set $\{q_1, \dots, q_{k-1}, q'_k, q_{k+1}, \dots, q_n\} - \{q_p\}$ is linearly independent, a contradiction. The result follows for the index p in like manner.

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